Time-Scale Synthesis of a Closed-Loop Discrete Optimal Control System

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A two-time-scale discrete control system is considered. The closed-loop optimal linear quadratic regulator for the system requires the solution of a full-order algebraic matrix Riccati equation. Alternatively, the original system is decomposed into reduced-order slow and fast subsystems. The closed-loop optimal control of the subsystems requires the solution of two algebraic matrix Riccati equations of an order lower than that required for the full-order system. A composite, closed-loop suboptimal control is created from the sum of the slow and fast feedback optimal controls. Numerical results obtained for an aircraft model show a very close agreement between the exact (optimal) solutions and computationally simpler composite (suboptimal) solutions. The main advantage of the method is the considerable reduction in the overall computational requirements for the closed-loop optimal control of digital flight systems.

Introduction

THE spread of reliable and inexpensive microcomputers has recently aroused considerable interest in digital control systems.[1-5] Digital flight control systems are known for their flexibility, reliability, and easy implementability.[6-10] The theory of singular perturbations and time scales with their twin advantages of order reduction and stiffness removal has been successful in the analysis and synthesis of continuous flight control systems.[11-15] However, the digital flight control systems having a two-time-scale character has not received any attention so far.

In this paper, we consider a two-time-scale discrete control system. The closed-loop optimal control of the system using linear quadratic (LQ) regulator theory involves the solution of the full-order algebraic matrix Riccati equation. On the other hand, we decompose the original system into slow and fast subsystems.[16-19] The closed-loop optimal control of the slow and fast subsystems leads to the solution of two reduced-order algebraic matrix Riccati equations. A composite, feedback, suboptimal control is composed out of the slow and fast feedback optimal controls.[20-22] The method is applied to a fifth-order digital aircraft model possessing three slow modes and two fast modes. The comparison of performance indices with the original optimal control and the composite suboptimal control are found to be in excellent agreement with the proposed method.

The highlight of the paper is that it is computationally much simpler to obtain the composite suboptimal control from the lower-order subsystems than to obtain the optimal control from the original higher-order system. This aspect of computational simplicity is believed to be a very desirable feature for the onboard real-time computation of adaptive feedback control laws for digital flight systems.

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Two-Time-Scale Discrete System

Consider a stable, linear, time-invariant discrete control system possessing a two-time-scale (slow and fast) character as[16-19]

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{1}$$

with initial conditions $x(k=0)=x(0)$; $z(k=0)=z(0)$, where, $x(k)$ and $z(k)$ are n- and m-dimensional slow and fast state variables, respectively; $u(k)$ is an r-dimensional control variable; and A_{ij} and B_i are submatrices of appropriate dimensionality.

The two-time-scale character of the system given by Eq. (1) implies that

$$\max |\lambda(A_f)| \ll \min |\lambda(A_s)| \tag{2}$$

where A_s and A_f are the slow and fast subsystem matrices whose expressions will be obtained below and

$$\lambda(A) = \lambda(A_s) U \lambda(A_f)$$

$$\lambda(A_s) = \{\lambda_{s1}, \lambda_{s2}, \ldots, \lambda_{sn}\}$$

$$\lambda(A_f) = \{\lambda_{f1}, \lambda_{f2}, \ldots, \lambda_{fm}\}$$

$$|\lambda_{s1}| \geq \ldots \geq |\lambda_{sn}| \gg |\lambda_{f1}| \geq \ldots \geq |\lambda_{fm}| \tag{3}$$

The original system of Eq. (1) is decoupled into slow and fast subsystems of n and m dimensions, respectively. The decoupling transformations to be used are given by[16,19]

$$\begin{bmatrix} x_s(k) \\ z_f(k) \end{bmatrix} = \begin{bmatrix} I_s + ED & E \\ D & I_f \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} \tag{4}$$

and

$$\begin{bmatrix} x(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} I_s & -E \\ -D & I_f + DE \end{bmatrix} \begin{bmatrix} x_s(k) \\ z_f(k) \end{bmatrix} \tag{5}$$

where $D(m\times n)$ and $E(n\times m)$ satisfy the algebraic Riccati type equations

$$A_{22}D-DA_{11}+DA_{12}D-A_{21}=0 \quad (6)$$

$$E(A_{22}+DA_{12})-(A_{11}-A_{12}D)E+A_{12}=0 \quad (7)$$

Let us note that the interesting feature of the decoupling transformations is that the inverse transformation given by Eq. (5) does not require any matrix inversions of D and E.

By using the transformation given by Eq. (4) in Eq. (1), we get the decoupled slow and fast subsystems,

$$x_s(k+1)=A_s x_s(k)+B_s u(k) \quad (8)$$

$$z_f(k+1)=A_f z_f(k)+B_f u(k) \quad (9)$$

where

$$A_s=A_{11}-A_{12}D; \qquad A_f=A_{22}+DA_{12}$$

$$B_s=(I_s+ED)B_1+EB_2; \quad B_f=DB_1+B_2$$

Because of the nonlinear nature of Eqs. (6) and (7), one has to obtain an iterative solution as[16]

$$D_{i+1}=[A_{22}D_i+D_iA_{12}D_i-A_{21}]A_{11}^{-1}$$

$$D_0=-A_{21}A_{11}^{-1} \quad (10)$$

$$E_{i+1}=A_{11}^{-1}[E_i(A_{22}+DA_{12})+A_{12}DE_i+A_{12}]$$

$$E_0=A_{11}^{-1}A_{12} \quad (11)$$

where we assume that A_{11} is nonsingular. We note that the subsystems given by Eqs. (8) and (9) have the advantage of mode decoupling.

Optimal Control of Original System and Subsystems

The performance index to be minimized with the original system of Eq. (1) is

$$J=\sum_{k=0}^{\infty}[y^T(k)Qy(k)+u^T(k)Ru(k)] \quad (12)$$

where

$$y^T(k)=[x^T(k),z^T(k)]$$

$$Q=\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22}\end{bmatrix}$$, a positive semidefinite symmetric matrix

R = a positive definite symmetric matrix

The closed-loop optimal control is given by[24]

$$u(k)=-R^{-1}B^TP[I+BR^{-1}B^TP]^{-1}Ay(k) \quad (13)$$

where P, of order $(n+m)\times(n+m)$, is the positive definite symmetric solution of the algebraic matric Riccati equation

$$P=A^TP[I+BR^{-1}B^TP]^{-1}A+Q \quad (14)$$

where

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}, \; B=\begin{bmatrix} B_1 \\ B_2\end{bmatrix}$$

The closed-loop optimal system is given by

$$y(k+1)=(A-BF)y(k) \quad (15)$$

where

$$F=R^{-1}B^TP[I+BR^{-1}B^TP]^{-1}A$$

Instead of tackling the original regulator problem described by Eqs. (1) and (12) directly, we decompose it appropriately into two regulator problems for slow and fast subsystems. The slow regulator problem consists of the slow subsystem of Eq. (8) and a quadratic performance index J_s. The fast regulator problem is composed of the fast subsystem of Eq. (9) and a quadratic performance measure J_f. We note that the subsystems measures are constructed in such a way that $J=J_s+J_f$.

Consider the performance index of Eq. (12) of the original system. Using the transformation of Eq. (5) in Eq. (12), it is easily shown that[20-22]

$$J_s=\sum_{k=0}^{\infty}[x_s^T(k)Q_sx_s(k)+u_s^T(k)Ru_s(k)] \quad (16)$$

and

$$J_f=\sum_{k=0}^{\infty}[z_f^T(k)Q_fz_f(k)+u_f^T(k)Ru_f(k)] \quad (17)$$

where the positive semidefinite, symmetric matrices Q_s and Q_f are given by

$$Q_s=Q_{11}+D^TQ_{22}D$$

$$Q_f=E^TQ_{11}E+(I_f+DE)^TQ_{22}(I_f+DE)$$

The slow regulator problem is to minimize J_s for the slow subsystem

$$x_s(k+1)=A_sx_s(k)+B_su_s(k) \quad (18)$$

The optimal feedback control of the slow subsystem is given by

$$u_s(k)=-R^{-1}B_s^TP_s[I_s+B_sR^{-1}B_s^TP_s]^{-1}A_sx_s(k) \quad (19)$$

where P_s, of order $n\times n$, is the positive definite symmetric solution of the reduced-order algebraic matrix Riccati equation,

$$P_s=A_s^TP_s[I_s+B_sR^{-1}B_s^TP_s]^{-1}A_s+Q_s \quad (20)$$

Similarly, the fast regulator problem is to minimize J_f for the fast subsystem

$$z_f(k+1)=A_fz_f(k)+B_fu_f(k) \quad (21)$$

The optimal feedback control of the fast subsystem becomes

$$u_f(k)=-R^{-1}B_f^TP_f[I_f+B_fR^{-1}B_f^TP_f]^{-1}A_fz_f(k) \quad (22)$$

where P_f, of order $(m\times m)$, is the positive definite symmetric solution of the reduced order algebraic matrix Riccati equation

$$P_f=A_f^TP_f[I_f+B_fR^{-1}B_f^TP_f]^{-1}A_f+Q_f \quad (23)$$

Rewrite the control laws given by Eqs. (19) and (22) as

$$u_s(k)=-F_sx_s(k) \quad (24)$$

$$u_f(k)=-F_fz_f(k) \quad (25)$$

where

$$F_s = R^{-1}B_s^T P_s [I_s + B_s R^{-1} B_s^T P_s]^{-1} A_s$$

$$F_f = R^{-1}B_f^T P_f [I_f + B_f R^{-1} B_f^T P_f]^{-1} A_f$$

We note that the control laws given by Eqs. (24) and (25) are optimal with respect to the slow and fast subsystems given by Eqs. (18) and (21) only. But, it is computationally simpler to determine these control laws than the optimal control law of Eq. (13) of the original system.

Composite Control

The composite control is formulated as the sum of the slow and fast feedback optimal controls given by Eqs. (24) and (25). That is,

$$u_c(k) = u_s(k) + u_f(k)$$

$$= -[F_s x_s(k) + F_f z_f(k)] \tag{26}$$

Now, using he transformation of Eq. (4) in Eq. (26), we get

$$u_c(k) = -[F_{sc}x(k) + F_{fc}z(k)]$$

$$= -F_c y(k) \tag{27}$$

where

$$F_{sc} = F_s(I_s + ED) + F_f D$$

$$F_{fc} = F_s E + F_f$$

$$F_c = [F_{sc}, F_{fc}]$$

Using the composite control of Eq. (27) in the original system of Eq. (1),

$$y_c(k+1) = (A - BF_c)y_c(k) \tag{28}$$

We know that evaluation of the original performance index of Eq. (12), assuming the composite control of Eq. (28), results in the suboptimal performance index

$$J_c = \tfrac{1}{2} y^T(0) P_c y(0) \tag{29}$$

where P_c is the positive definite symmetric solution of the discrete Lyapunov equation,[22,25]

$$P_c = (A - BF_c)^T P_c (A - BF_c) + Q + F_c^T R F_c \tag{30}$$

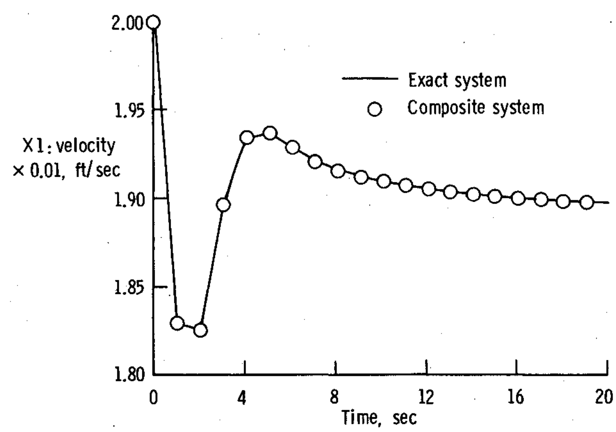

Fig. 1 Comparison of solutions of velocity.

Numerical Results for an Aircraft Model

The longitudinal dynamic equations for the aircraft model are obtained based on a rigid-body assumption.[23] Also, the angle of attack is taken to be small. It has been observed that the linearized model possesses a two-time-scale property in the sense that the pitch angle, velocity, and altitude are the "slow" variables and the angle of attack and pitch rate are the "fast" variables. For digital implementation, we obtain the discrete model using a zero-order hold. For the aircraft under consideration the discrete model is given by[19,23]

$$y(k+1) = Ay(k) + Bu(k)$$

where $x_1(k)$ = velocity, ft/s; $x_2(k)$ = pitch angle, deg; $x_3(k)$ = altitude, ft; $z_1(k)$ = angle of attack, deg;

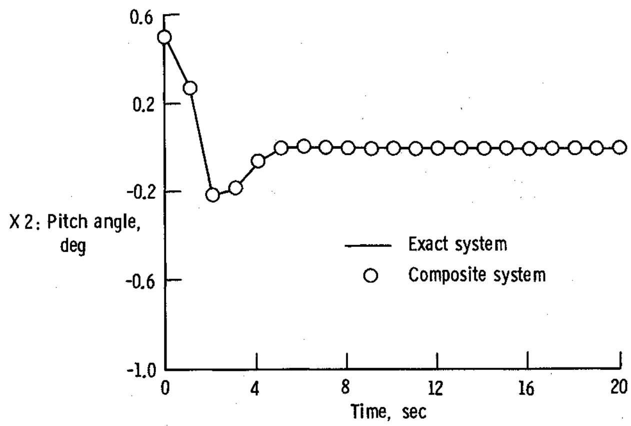

Fig. 2 Comparison of solutions of pitch angle.

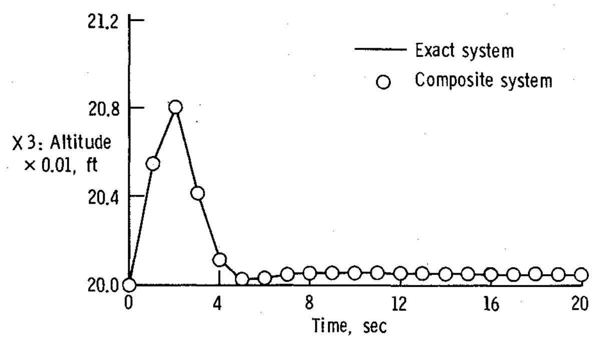

Fig. 3 Comparison of solutions of altitude.

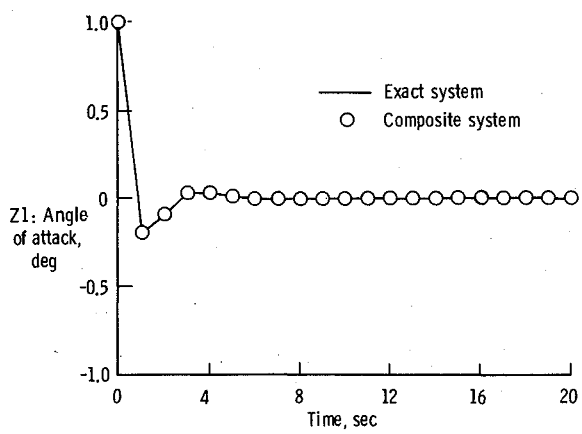

Fig. 4 Comparison of solutions of angle of attack.

$z_2(k)$ = pitch angular velocity, deg/s; $u_1(k)$ = elevator deflection, deg; $u_2(k)$ = flap deflection, deg; $u_3(k)$ = throttle position, deg; and

$$A=\begin{bmatrix} 0.923701 & -0.308096 & 0.000000 & 0.053043 & -0.090367 \\ 0.039705 & 0.995525 & 0.000000 & -0.107454 & -0.588883 \\ 0.087127 & 1.899490 & 1.000000 & -0.635270 & 0.394015 \\ -0.035537 & 0.010123 & 0.000000 & 0.007748 & 0.137407 \\ 0.069562 & -0.012706 & 0.000000 & -0.097108 & 0.287411 \end{bmatrix} \qquad B=\begin{bmatrix} 0.042825 & -0.000395 & -0.154048 \\ -0.484628 & -0.515349 & -0.002237 \\ -0.161525 & -0.067522 & -0.005257 \\ -0.202010 & -0.289303 & 0.005061 \\ -0.806770 & -0.852161 & -0.006353 \end{bmatrix}$$

The states $x_1(k)$ and $x_3(k)$ are scaled down by a factor of 100 to facilitate easy representation.

The eigenvalues of the discrete model are 1.0, $0.962103 \pm j0.175343$, 0.217298, and 0.072881, indicating that there are three $[x_1(k), x_2(k)$, and $x_3(k)]$ "slow" states and two $[z_1(k)$ and $z_2(k)]$ "fast" states with eigenvalue separation ratio of 4.5005.

The various performance measures in Eq. (12) are taken as

$$Q_{11} = Q_{22} = I; \qquad R = r = 1$$

Using the method described in previous sections, the results obtained are summarized below.

The positive definite matrix P of Eq. (14) is

$$P=\begin{bmatrix} 4.822583 & 0.427518 & 0.709917 & -0.272287 & 0.059216 \\ 0.427518 & 11.654952 & 3.732418 & -2.665579 & 3.042291 \\ 0.709917 & 3.732418 & 2.690923 & -1.096067 & 0.882337 \\ -0.272287 & -2.665579 & -1.096067 & 1.742674 & -0.689301 \\ 0.059216 & 3.042291 & 0.882337 & -0.689301 & 2.015052 \end{bmatrix}$$

Using P, the optimal control of Eq. (13) is given as

$$F=\begin{bmatrix} -0.021105 & -1.136159 & -0.268959 & 0.260873 & -0.507318 \\ -0.110915 & -0.700095 & -0.101750 & 0.142104 & -0.410844 \\ -0.629305 & -0.025190 & -0.100651 & 0.033007 & 0.003334 \end{bmatrix}$$

The closed-loop optimal system of Eq. (15) has the eigenvalues 0.843545, $0.25879 \pm j0.326649$, and $0.06457 \pm j\,0.047911$.

For the slow and fast subsystems, the corresponding values are

$$P_s=\begin{bmatrix} 4.621454 & 1.051044 & 0.945863 \\ 1.051044 & 13.073895 & 4.463674 \\ 0.945863 & 4.463674 & 2.710497 \end{bmatrix} \quad \text{and} \quad P_f=\begin{bmatrix} 1.204794 & 0.244345 \\ 0.244345 & 1.914426 \end{bmatrix}$$

For the composite control, the feedback matrix F_c is obtained as

$$F_c=\begin{bmatrix} -0.010457 & -1.138702 & -0.270112 & 0.262312 & -0.507373 \\ -0.110238 & -0.699239 & -0.101678 & 0.142112 & -0.410528 \\ -0.654236 & -0.018681 & -0.100062 & 0.031065 & 0.004602 \end{bmatrix}$$

and the positive definite matrix P_c, obtained from the discrete Lyapunov equation (30), is

$$P_c=\begin{bmatrix} 5.045733 & 0.338736 & 0.671223 & -0.205383 & -0.013507 \\ 0.338736 & 11.734399 & 3.769142 & -2.716177 & 3.103791 \\ 0.671223 & 3.769142 & 2.714962 & -1.128617 & 0.918364 \\ -0.205383 & -2.716177 & -1.128617 & 1.790493 & -0.744196 \\ -0.013507 & 3.103791 & 0.918364 & -0.744196 & 2.083318 \end{bmatrix}$$

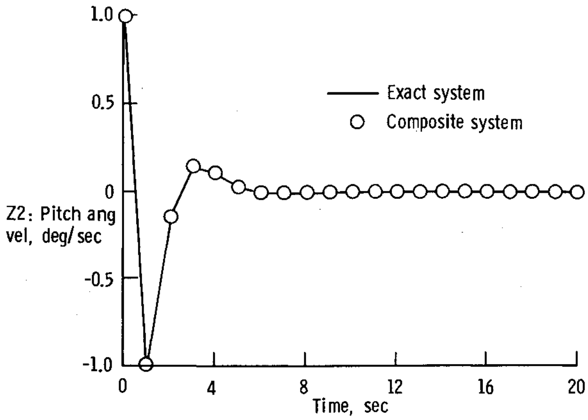

Fig. 5 Comparison of solutions of pitch angular velocity.

The corresponding closed-loop suboptimal composite system has the eigenvalues 0.840132, $0.259351 \pm j0.334107$, and $0.063358 \pm j0.04656$. The performance indices of the original optimal system and the composite suboptimal system are obtained as 612.1122 and 616.1734, respectively, with a percentage error of 0.66347. The central processing time on the computer required for the original system is 5.42 s, whereas that required for the composite system is only 0.44 s.

The responses of the various states for the exact (optimal) system and the composite (suboptimal) system are shown in Figs. 1–5. These responses are obtained for the case when the aircraft trajectories are to be regulated to the equilibrium flight conditions of $x_{1e} = 190.66$ ft/s, $x_{2e} = 0$, $x_{3e} = 2000$ ft, $z_{1e} = 0$, and $z_{2e} = 0$ with equilibrium controls.

The above results clearly show an excellent agreement between the exact system and the composite system. Once again, we note that the composite control is obtained from the low-order slow and fast subsystems with a considerable reduction in the overall computation.

Conclusions

We have addresed a two-time-scale discrete system. The closed-loop optimal control of the system required the solution of the high-order algebraic matrix Riccati equation. Alternatively, the original (exact) system was decomposed into low-order slow and fast subsystems. The feedback optimal control of the subsystems needed the solution of two reduced-order algebraic matrix Riccati equations. A composite, closed-loop, suboptimal control has been formed out of the slow and fast feedback optimal controls.

A digital flight control system has been analyzed using the method. The fifth-order discrete model has the pitch angle, velocity, and altitude as the slow variables and the angle of attack and pitch rate as the fast variables. The numerical results have shown an excellent agreement between the exact (original) system and composite (suboptimal) system.

The main advantage of the method is the computational simplicity in obtaining the composite, near-optimal, feedback control from the low-order subsystems as an alternative to the computationally expensive optimal feedback control from the original high-order system. This inexpensive computational reduction is an essential feature for the onboard real-time computation of adaptive feedback control laws for digital flight systems.

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